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Jan von Plato, *Elements of Logical Reasoning*, Cambridge University Press, 2013, 264 pp., ISBN 978-1-107-61077-4.

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Elements of Logical Reasoning by Jan von Plato is a welcome addition to textbooks on logic. It covers both introductory and more advanced topics sprinkled throughout with points of philosophical and historical interest. It is a refreshing introduction to the subject from the point of view of proof theory. I therefore fully endorse this book. Its main object of study are deductions in a formal language. This is in contrast to most textbooks, which take models or valuations to be the primary object of logical study. Intuitionistic logic is better suited to a proof-theoretic setting than classical logic. Because of certain features of intuitionistic negation finding the proof of a proposition requires less guesswork than classical proof of a proposition might. Intuitionistic logic differs from classical logic in its denial of the logical truth of the law of excluded middle, i.e. the claim that every sentence is either true or false. This approach to the subject will be fruitful for students coming to logic for the first time and for those who are interested in non-classical logics.

The book is divided into four parts: First Steps in Logical Reasoning, Logical Reasoning with Quantifiers, Beyond Pure Logic, and Complementary Topics. The number of chapters per part ranges from two to seven, the first part being the longest.

Chapter one is a discussion of inference and deduction without any formalization. The second and third chapters succinctly and clearly introduce a formal language whose logic is the study for the rest of the chapter. A formal language is a set of sentences constructed by an inductive procedure. The base step adds an infinite number of sentences which have no sentences as parts, called atomics. The expressions that are being studied are ones that produce sentences from sentences: a conditional (→), a negation (¬), a conjunction (∧), and a disjunction (∨). For any two sentences of the language A and B, ($A \rightarrow B$), ($\neg A$), ($A \wedge B$), and ($A \vee B$) are all also sentences of the language. This definition of a formal language is standard and it or some variant of it would appear in any logic textbook. These correspond roughly to the English sentences 'If *A* then *B*', 'It is not the case that *A*', '*A* and *B*', and '*A* or *B*' respectively. Nothing else gets to be a sentence except by being atomic or through the above condition.

A logic is a set of sentences, the set of sentences that are logically true with respect to a language. A calculus is a set of inferences that generate a logic for a language. A logic for the formal language introduced above is generated by saying which sentences can be inferred from which others, i.e. specifying rules of inference for sentences of the language. A small set of inferences are taken as unjustified and other good inferences are justified in terms of that small set. For instance, the rules governing any sentence of the form $A \wedge B$ are that if *A* and *B* are true, then $A \wedge B$ is true and if $A \wedge B$ is true, then *A* is true and *B* is true. The first of these clauses is called the introduction rule for conjunction (∧) the second is called its *elimination rule*. This is of philosophical interest because the meaning of ∧ is determined by these rules. The same holds for any of the other expressions of the language. The thesis that the meaning of an expression is at least determined by the contribution it makes to good inferences is called inferentialism. If inferentialism is true, then notions like reference and truth play a secondary role in a semantic theory while inference and validity take the spotlight.

The inferentialist theory advocated by von Plato takes the introduction rules for an expression to be primary for determining the meaning of that expression. He offers an explanation of how the elimination rules can be in some sense 'derived' form the introduction rules. This is a rich and fertile idea that began with Gentzen and saw further development in the work of Prawitz (see *Remarks on Some Approaches to the Concept of Logical Consequence*, Synthese (1985)) and Dummett (see *The Logical Basis of Metaphysics*). This view is referred to by Dummett as *justificationism*.

A deduction of a sentence *S* from a set of sentences Γ is a set of sentences arranged so that each sentence follows via an introduction or elimination rule from the sentences directly above it, is an element of Γ that features as an assumption, and of which *S* is the last sentence. A deduction is normal iff all the major premises of elimination rules occur as assumptions. A set of inference rules for a language is normalizable when for any deduction, normal or otherwise, of a sentence there is a normal deduction of that sentence. These definitions are relatively standard. As von Plato presents, these notions would be adequate for exploring natural deduction systems as they are presented in other textbooks. Taking justificationism as the starting point explains the emphasis of normalization in the book. In a normal deduction the only applications of elimination rules are to sentences that have not been generated by introduction rules, all of the real work is done by the introduction rules. On a justificationist picture of meaning normalization explains how the introduction rules are semantically primary.

In addition to being philosophically important normalization results entail that the logic in question has other interesting properties. It is therefore fitting that von Plato gives normalization results a key role to play throughout the textbook. In a normal deduction every sentence is either a sub-sentence of an open formula or the conclusion of the deduction. This is a useful feature of a calculus for computation. If a computer were trying to search for a deduction of a sentence it would only have to search that sentence's sub-sentences or potential open sentences in the deduction. Computationally this makes a logic much more manageable. In intuitionistic logic a normal deduction that ends in a sentence of the form $A \vee B$ ends with a rule of disjunction introduction, i.e. If *A* is true or *B* is true, then $A \vee B$ is true. It follows that if $A \vee B$ is a logical truth, then either *A* is a logical truth or *B* is a l' here append 'ogical truth' to 'l'. This means that proof-search is simplied when the conclusion of the deduction in question is a disjuction. From a philosophical perspective it is a guarantee that the normal deduction of a disjunction does not rely on rules governing any other connectives. This nice feature is distinctive of intuitionistic logic. It fails for classical logic.

A sequent calculus is another way to present a logic. Von Plato introduces sequent calculi in the service of better understanding natural deduction systems. While this is appropriate for this textbook it is worth mentioning that sequent calculi are of great proof-theoretic interest in their own right. Instead of inferences moving from sentences to a sentence, as in a natural deduction calculus, in a sequent calculus inferences move from sequents to a sequent. A sequent is a set of sentences one of which is the conclusion and the rest of which are the premises. Sequent calculi are helpful in proof search because a sequent contains not only the formula to be proved but the open sentences or assumptions on which it relies. Classical logic is better suited to a sequent calculus than a natural deduction calculus. A particularly positive feature of von Plato's presentation of this material is that it makes this clear by presenting natural deduction calculi for both intuitionistic and classical logic first. Only later does he present sequent calculi for both logics. It is clear in the natural deduction setting the elimination rules for negation are not appropriate given its introduction rule.

The last chapter of Part I includes a discussion of what are commonly called the semantics of propositional logics. It describes truth tables for classical logic and presents a concise description of Kripke semantics for intuitionistic logic. It also points out some oddities that occur when the classical conditional is combined with classical disjunction. For instance, $(A \rightarrow B) \vee (B \rightarrow A)$ is a classical logical truth. Von Plato calls this 'Dummett's Law' and draws attention to it in order to pose a problem for classical logic. It is not however clear that this is a mark against classical logic. Whether or not this is a plausible logical truth depends on what reading of logical truth is appropriate to the logic in question. Each logic comes with a different understanding of what the sentences of its language mean. In intuitionistic logic the most natural reading of the sentence $A \rightarrow B$ is as saying that there is a transformation of a deduction of A to a deduction of B. Given this reading Dummett's Law says that for any two sentences there is a way of transforming a deduction of one into a deduction of the other. This is implausible. Take the two sentences 'It is raining' and 'Kangaroos are mammals'. There does not seem to be any way to transform a deduction of either one into a deduction of the other. If the most natural reading of $A \rightarrow B$ for a logic is as saying that there is a transformation of a deduction of A into a deduction of B, then Dummett's Law ought to fail. That reading is thus not appropriate for classical logic given its logical truths. Suppose that $\Gamma \vdash A$ is a valid classical argument. The most natural reading of that fact is that it is impossible to make all of Γ true and make A false. In a case where Γ is empty this means that there is no way to make A false. To put the point in other words if A is a classical tautology, then there is no way to make A false. Thus, this is different from the intuitionistic reading of logical truth. Since Dummett's Law is logically true according to classical logic the intuitionist reading of $A \to B$ sketched above cannot be the one appropriate for classical logic. The classical reading of what it is for a sentence to be logically true makes Dummett's Law more plausible.[1] According to the classical reading, that Dummett's Law is a logical truth says that it is impossible to make $(A \to B) \vee (B \to A)$ false. If that's the case, then there had better be no way of making both $A \to B$ and $B \to A$ false. To establish that there is no way of doing this suppose that there were. In order to make a sentence of the form $\varphi \to \psi$ false it must be that φ is true and ψ is false. Under the above supposition A would be true and B false and B true and A false. But that is clearly impossible, so there is no way to make $(A \to B) \vee (B \to A)$ false. Once the appropriate reading is given to each logic it is possible to see why Dummett's Law is not an intuitionistic logical truth while it is a classical logical truth.

Part I concludes with a philosophical discussion of the history and philosophy of logic. Of particular interest is von Plato's discussion of the difficult question of whether truth is conceptually prior to proof or proof conceptually prior to truth. Intuitionistic logic is well-paired with a philosophy that takes the laer route. The above reading suggests that intuitionistic connectives are best read as directions for transforming deductions into a deduction. Classical connectives are best read as stating relations between the truth and falsity of sentences. In this way classical logic takes truth to be primary. A proof is a guarantee that the premises of an argument cannot be true while the conclusion is false. Von Plato's discussion of this is brief but he brings the reader into close contact with some of the most difficult and interesting philosophical questions about what the world is like and the relationship of knowers to the world. Different logics answer those questions differently. Von Plato sums up the role of the study of logic in answering those philosophical questions by saying that it is "epistemology in laboratory settings".

Part II introduces predicate logic. Predicate logic studies the expressions, $\forall$ and $\exists$ called the Universal and the existential quantifiers. These

[1] In fact, an intuitionist agrees with this claim. A sentence A is intuitionistically impossible to make false just when $\neg\neg A$ is true. But $\neg\neg((A \to B) \vee (B \to A))$ is an intuitionistic logical truth.

are roughly translated as 'for all' and 'there is' respectively. Von Plato helpfully cites two explanations for the meaning of the universal quantifier. He attributes to Tarski the view that a sentence of the form $\forall x\varphi$ is true in a domain of objects iff φ is true of each element of the domain. One of the problems that he cites with this definition is that it presupposes an antecedent understanding of what a domain of objects is. The alternative account which von Plato attributes to Frege and Gentzen is that a sentence $\forall x\varphi$ is provable iff φ with y substituted for x is provable for an arbitrary y. As stated the above account does not obviously require an antecedent grasp of a domain of objects, though in the case where y is a name this is less certain. This discussion is again related to the question of whether truth is conceptually prior to proof or vice versa. The Tarskian view appears to take truth as primitive in this order while the Frege-Gentzen view takes proof as primitive. While this distinction may be helpful for some philosophical purposes, it should be noted that what exactly the views of Frege, Gentzen, and Tarski on quantification are is controversial.

Part II contains proofs of some interesting features of natural deduction and sequent calculi for both intuitionistic and classical logic. It concludes with a discussion of the semantics of quantified logic. A rough description of the model theory for first-order logic is given. Of particular interest is that the discussion in this part proceeds without mention of the notion of "set" or other tools that are commonly employed in model theory. In fact, most of the proofs done with models are done in a proof theoretic metalanguage. This is entirely appropriate for the book, deductions are the primary object of study. The notion of a Kripke model for first-order intuitionistic logic is presented in an equally succinct way. The main point of presenting these is to bring to light the difference between classical and intuitionistic accounts of quantification. It is noteworthy how well von Plato accomplishes this task. He points out that intuitionistic domains of quantification expand as more entities are discovered. Classical domains of quantification are static. The intuitionistic universal quantifier rangers over all the expansions of the domain. This is not the same as ranging over a static domain of entities as the classical quantifier does.

The final two parts of the book deal with more advanced topics in proof theory. Part III covers identity and number theory and Part IV covers normalization and cut elimination.

Identity is introduced first into the natural deduction calculus for intuitionistic logic via axioms. These are shown to be equivalent to a set of rules that define the identity relation. Given that one of the main themes of the book is inferentialism of the sort discussed above this is a fitting approach to offering an account of identity. After a discussion of various attempts at defining identity there is an explanation of sense and reference. Since this is a philosophical review it is only appropriate that some philosophical dispute is dealt with. The sense of an expression is said to be the way that it is built. For instance the sense of '3 x 6' is given by the sense of multiplication and the senses of '3' and '6'. The reference of that expression is the reference of '18', whatever that may be. This explanation of sense is helpful for complex terms but it does not immediately provide an account of the sense of expressions which are not built up out of anything else. Von Plato uses a geometrical example to state in more detail what the sense of an expressions is. Let $p(x, y)$ be a function that denotes the line parallel to x that intersects y. Let l and m be lines and a a point. The function $p(x, y)$ allows us to construct other lines, $p(l, a)$ and $p(m, a)$. Suppose further that m and l are parallel. By the Euclidean axiom of parallels, the line $p(l, a)$ coincides completely with the line $p(m, a)$. The senses of the expressions '$p(l, a)$' and '$p(m, a)$' are different even though they refer to the same line. Von Plato suggests that identity is identity of construction. This is a revisionary use of the term 'identity'.[2] While this may work as an account of the sense of some descriptive terms it does not immediately suggest an explanation of how expressions like 'is a car' or 'Aristotle' come to have a sense or how their senses might be identified. Following the geometrical discussion von Plato suggests that if an identity is true, then it should be immediately recognizable that it is true. This is a bold claim that is not in step with much of contemporary metaphysics. Neither of those count against the truth of that claim but leave one wanting more explanation. This is, of course, a minor point that in no way detracts from the main thrust of the text.

Part III concludes with a discussion of the Peano axioms and how they may be added to the existing systems of natural deduction. Again axioms

[2] Although there may be an antecedent to this use of the term 'identity' in one interpretation of Frege's account of sense.

are transformed into rules of inference. Several different accounts of the natural numbers are given including Robinson Arithmetic and Heyting Arithmetic. The first chapter of Part IV is the most challenging of the book. The topics covered in this section of the book will be surveyed only, with the details left to the side. It may be helpful to some readers to remark that this material could be taught in an upper level undergraduate class or a beginning graduate class in proof-theory. It introduces in a cogent way the mechanism that governs inductive proofs. It also covers a proof of normalization for the natural deduction calculus for intuitionistic logic. Following this is a discussion of the Curry-Howard isomorphism according to which there is a precise correspondence between certain programs in a computational setting and proofs in a mathematical setting. A cut elimination theorem for intuitionistic logic is proved by means of a correspondence between normal deductions and cut-free deductions.

The book concludes with a brief history of deductive logics beginning with Aristotle and continuing through to Heyting. It traces the notion of a syllogism from Aristotle to Boole. Boole's work made it possible to represent syllogisms mathematically and to offer a mathematical treatment of hypothetical propositions. A nice introductory discussion of the history of algebraic approaches to logic is presented and connected to the axiomatic approaches of Frege and later Whitehead and Russell. Von Plato conjectures that Heyting's axioms for intuitionistic logic were the kernel that lead to the growth of Gentzen's systems of natural deduction and sequent calculi.

This is an excellent advanced textbook in logic. It could be easily adapted to guide an upper level undergraduate or first year graduate course in logic. The topics covered at the beginning are introductory enough that students who have not seen proof theoretic methods in logic before – or any logic at all – will be able to grasp the material. The book concludes with proofs that are appropriate for an advanced class in logic. Sprinkled throughout are interesting philosophical discussions of proof and truth and their relation to intuitionistic and classical logic.